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Moments of lepton spectrum in B decays and the $m_b - m_c$ quark mass difference.

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Abstract

It is argued that the quark mass difference $m_b - m_c$ can be extracted with a high accuracy from experimental data on ratia of moments of lepton energy spectrum in semileptonic decays of B mesons. Theoretical expressions for the moments are presented, which include perturbative as well as non-perturbative corrections.

1 Introduction

Masses of quarks as well as weak mixing angles are fundamental input parameters in the Standard Model. Thus it is important to know them with maximal possible accuracy. Moreover, the values of the masses of the b and c quarks are correlated with the determination of the mixing parameter $|V_{bc}|$ from the data on inclusive semileptonic B meson decay rates. Therefore an independent understanding of m_b and m_c is necessary for a precision determination of $|V_{bc}|$. There is a vast literature on extracting the values of m_c and m_b from the data on charmonium, Υ resonances, charmed and B hadrons. However there is hardly a compelling argument in the literature to invalidate the original evaluations from the QCD sum rules of the 'on shell' quark masses: $m_c = 1.35 \pm 0.05\, GeV$ [1, 2, 3] and $m_b = 4.80 \pm 0.03\, GeV$ [4], which can still serve as 'reference values' in discussion of dynamics of heavy hadrons. It is the purpose of this paper to point out that the accuracy of determining the difference between the quark masses $m_b - m_c$ can possibly be significantly improved by considering the ratia of the moments

$$M_n = \int E_l^n \frac{d\Gamma}{dE_l}\, dE_l \tag{1}$$

of the lepton energy spectrum $d\Gamma/dE_l$ in semileptonic B decays. While theoretical expressions for the moments are sensitive to combinations of m_b and m_c (and also to $|V_{bc}|$) , their ratia for few first moments are to a high accuracy sensitive only to the mass difference $m_b - m_c$. Also it is natural to expect that experimentally the ratia of the moments are determined with better systematic accuracy, since the absolute normalization of the event rate cancels out in the ratia.

On the theoretical side the ratia of the moments have the advantage of weak and controlled dependence on the infrared dynamics in QCD both perturbatively and non-perturbatively. For first few moments the perturbative corrections are expressed through $\alpha_s(m_b)$ and the non-perturbative ones are suppressed by m_b^{-2} and can be found by the OPE technique[5, 6, 7, 8] in terms of the quantities μ_π^2/m_b^2 and μ_g^2/m_b^2 with

$$\mu_\pi^2 = \langle B|(\bar{b}\boldsymbol{\pi}^2 b)|B\rangle \qquad \text{and} \qquad \mu_g^2 = \langle B| \left(\bar{b}\,(\boldsymbol{\sigma}\cdot\boldsymbol{B})\, b\right)|B\rangle\ , \tag{2}$$

where B is the chromo - magnetic field operator and $\boldsymbol{\pi} = \mathbf{p} - \mathbf{A}$ is the covariant momentum operator for the heavy quark. The spin-dependent chromo - magnetic energy μ_g^2 is related to the mass splitting of B^* and B mesons: $\mu_g^2 = \frac{3}{4}\,(M_{B^*}^2 - M_B^2) \approx 0.36\, GeV^2$, while for the kinetic energy μ_π^2 only a lower bound exists[9]: $\mu_\pi^2 \geq \mu_g^2$, which follows from non-negativity of the operator $(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2 = \boldsymbol{\pi}^2 - \boldsymbol{\sigma}\cdot\boldsymbol{B}$ and an estimate[10] $\mu_\pi^2 \approx 0.5 \pm 0.1\, GeV^2$ from QCD

sum rules.

It should be also noticed that the difference $m_b - m_c$, unlike each of the masses, is less sensitive to the infrared behavior in QCD and is a well defined quantity in QCD in the limit where both masses are heavy as compared to Λ_{QCD}. Indeed, because of confinement there is no real 'mass shell' for a quark. Therefore its mass can only be determined off shell and then extrapolated to a would-be on-shell value. For a heavy quark its mass can be found at a virtuality scale μ (i.e. at $m^2 - p^2 \approx 2\,m\,\mu$) such that on one hand $\mu \gg \Lambda_{QCD}$, which justifies a short-distance treatment, and on the other hand $\mu \ll m$. The latter condition ensures that the evolution of $m(\mu)$ towards the would-be mass shell does not depend on m in the leading order in $1/m$. In particular, in the leading log approximation this evolution is described by the RG equation[11, 12]

$$\frac{d\,m}{d\mu} = -c\,\alpha_s(\mu) \ , \tag{3}$$

where the constant c depends on the specific definition of the off-shell mass. The infrared singularity of α_s (infrared renormalon) prevents from integrating an equation like (3) down to $\mu = 0$ and thus really extrapolating the mass to the mass shell. However one can integrate the equation (3) in any finite order in α_s and thus define the 'on-shell' mass of a heavy quark to a finite order. In this sense the 'on-shell' masses m_b and m_c quoted above are the result of such extrapolation in the first order and are thus appropriate for using in other calculations in the first order in α_s. Naturally this definition of quark mass changes with the order in α_s. However, increasing the order in α_s does not converge at a certain value of m because of the factorial divergence of the series in α_s caused by the infrared renormalon. A minimal residual error in the 'on shell' mass in this procedure is of the order of Λ_{QCD} [12]. On the other hand this uncertainty in a heavy quark mass does not depend on m in the limit of large m. Thus this uncertainty cancels in the difference of masses of two heavy quarks. As to the pre-asymptotic in the heavy quark mass limit corrections to the evolution equation (3), their contribution to the residual uncertainty is of the order of Λ_{QCD}^3/m^2, which is quite small even for the charmed quark.

The difference $m_b - m_c$ can be estimated from the experimental values of the masses of D and B mesons:

$$M_B - M_D = m_b - m_c + \frac{\mu_\pi^2 - \mu_g^2}{2\,m_b} - \frac{\mu_\pi^2 - \mu_g^2}{2\,m_c} + o(m_c^{-2},\, m_b^{-2}) \tag{4}$$

Neglecting the terms, smaller than m_c^{-2} or m_b^{-2}, and taking into account the inequality

$\mu_\pi^2 \geq \mu_g^2$ one finds a lower bound for the difference of the quark masses:

$$m_b - m_c \geq M_B - M_D = 3.41\, GeV \ . \tag{5}$$

Varying μ_π^2 in the range from $\mu_\pi^2 = \mu_g^2 \approx 0.36\, GeV^2$ up to $\mu_\pi^2 = 0.6\, GeV^2$ one finds $m_b - m_c = 3.44 \pm 0.03\, GeV$, which is perfectly compatible with the quoted above estimates of each of the quark masses from QCD sum rules.

Naturally, an independent measurement of this quark mass difference with a comparable or better accuracy would provide an additional consistency check for the heavy quark theory and, possibly, would enable a better quantitative understanding of the parameter μ_π^2. As is discussed in the rest of this paper, a measurement of the ratia of few first moments (1) provides an excellent opportunity to independently determine the difference $m_b - m_c$.

2 Moments of the lepton spectrum

In the simplest approximation, where the QCD effects are neglected altogether the spectrum of charged lepton l in the decay $b \to c\, l\, \nu$ is given by the well-known muon decay formula

$$\frac{d\,\Gamma}{dx} = \Gamma_0\, w_0(x,\, \mu) \tag{6}$$

with $\Gamma_0 = G_F^2\, |V_{cb}|^2\, m_b^5/(192\, \pi^3)$ and

$$w_0(x,\, \mu) = \frac{2\, x^2\, (1 - \mu^2 - x)^2}{(1-x)^3} \Big[(1-x)\,(3-2x) + \mu^2\,(3-x) \Big] \ , \tag{7}$$

where $\mu = m_c/m_b$ and $x = 2\, E_l/m_b$, so that the physical range of x goes from $x_m = 0$ to $x_M = 1 - \mu^2$.

With the first perturbative QCD correction and the first non-perturbative corrections, proportional to μ_π^2/m^2 and μ_g^2/m^2, taken into account the formula for the differential decay rate can be written as

$$\frac{1}{\Gamma_0}\, \frac{d\,\Gamma}{dx} = w_0(x,\, \mu) - \frac{2}{3}\, \frac{\alpha_s}{\pi}\, w_1(x,\, \mu) + \frac{\mu_\pi^2}{m_b^2}\, w_\pi(x, \mu) + \frac{\mu_g^2}{m_b^2}\, w_g(x, \mu) \tag{8}$$

where the explicit expression for the perturbative correction function $w_1(x,\, \mu)$ is extremely lengthy and can be found in the original papers[13, 14] (see also [15, 16])[1]. The non-

[1] I am thankful to M. Ježabek, for pointing out to me the papers [13] and [14], where the calculation of the function $w_1(x,\, \mu)$ has been finalized, and for sending me his and A. Czarnecki's FORTRAN code for numerical calculation of this function.

perturbative correction functions $w_\pi(x,\,\mu)$ and $w_g(x,\,\mu)$ are given by[7, 8]

$$w_\pi(x,\,\mu)=\frac{2\,x^3}{3\,(1-x)^5}\quad\left(-5-15\,\mu^4+20\,\mu^6+25\,x+21\,\mu^4\,x-10\,\mu^6\,x-50\,x^2-\right.$$
$$\left.6\,\mu^4\,x^2+2\,\mu^6\,x^2+50\,x^3-25\,x^4+5\,x^5\right)\ , \tag{9}$$

$$w_g(x,\,\mu)=\frac{2\,x^2\,\left(1-\mu^2-x\right)}{3\,(1-x)^4}\quad\left(6-12\,\mu^2-30\,\mu^4-13\,x+23\,\mu^2\,x+20\,\mu^4\,x+3\,x^2-\right.$$
$$\left.16\,\mu^2\,x^2-5\,\mu^4\,x^2+9\,x^3+5\,\mu^2\,x^3-5\,x^4\right)\ . \tag{10}$$

The relative correction $w_1(x,\,\mu)/w_0(x,\,\mu)$ has a logarithmic singularity at the upper endpoint $x=x_M$ of the spectrum[15], which is a usual consequence of the Sudakov formfactor[17]. The relative non-perturbative corrections are still more singular: the ratio $w_g(x,\,\mu)/w_0(x,\,\mu)$ has a pole at $x=x_M$ and the ratio $w_\pi(x,\,\mu)/w_0(x,\,\mu)$ has a double pole at the upper endpoint, which in particular reflects the difference in the kinematics of decay of a free heavy quark and of a heavy quark bound in hadron[7, 18]. This implies that the spectrum close to the endpoint is sensitive to the infrared hadron dynamics, while in integral quantities, like the moments of the lepton spectrum, the effects of this dynamics are integrated over and are present only in the form of small corrections. Naturally, this conclusion is valid only if the number n of the moment is not parametrically large, since high moments measure the spectrum near the upper endpoint, and all the infrared effects come back. In the expressions for the moments this growth of sensitivity to large distances reveals itself in the growth with n of the relative magnitude of the non-perturbative corrections. Therefore, one can consider as 'safe' the moments with such n, for which the non-perturbative correction is still small. As will be discussed, if one chooses to keep individual terms in the corrections at a level below 10% - 15%, this would limit the range of n to $n\le 5$. Thus in what follows explicit results are presented for the moments in this range of n.

According to eq.(8) ratio of the n-th moment to M_0 (the total rate) $r_n=M_n/M_0$ can be written as

$$r_n=r_n^{(0)}\left(1-\frac{2}{3}\frac{\alpha_s}{\pi}\,\delta_n^{(1)}+\frac{\mu_\pi^2}{m_b^2}\,\delta_n^{(\pi)}+\frac{\mu_g^2}{m_b^2}\,\delta_n^{(g)}\right)\ , \tag{11}$$

where $r_n^{(0)}$ is the same ratio in the lowest approximation:

$$r_n^{(0)}=\left(\frac{m_b}{2}\right)^n\frac{\int_0^{1-\mu^2}w_0(x,\,\mu)\,x^n\,dx}{\int_0^{1-\mu^2}w_0(x,\,\mu)\,dx} \tag{12}$$

and the corrections $\delta_n^{(1)}$, $\delta_n^{(\pi)}$ and $\delta_n^{(g)}$ each being a function of μ are obtained from integrals with the corresponding correction function $w(x,\,\mu)$ in eq.(8) as

$$\delta_n = \frac{\int_0^{1-\mu^2} w(x,\,\mu)\, x^n\, dx}{\int_0^{1-\mu^2} w_0(x,\,\mu)\, x^n\, dx} - \frac{\int_0^{1-\mu^2} w(x,\,\mu)\, dx}{\int_0^{1-\mu^2} w_0(x,\,\mu)\, dx}\ . \tag{13}$$

The moments of the lowest order function $w_0(x,\,\mu)$ for $n \le 5$ are listed in the Appendix. The correction coefficients $\delta_n^{(\pi)}$ can in fact be found in a simple analytical form. This is possible due to the fact that the function $w_\pi(x,\,\mu)$ is related to a modification by a small boost with $< \mathbf{v}^2 >= \mu_\pi^2/m_b^2$ of the lepton spectrum described by the function $w_0(x,\,\mu)$ [7, 18]:

$$w_1(x,\,\mu) = \frac{x^2}{2}\frac{\partial}{\partial x}\left(\frac{w_0(x,\,\mu)}{x}\right) + \frac{x^3}{6}\frac{\partial^2}{\partial x^2}\left(\frac{w_0(x,\,\mu)}{x}\right) - \frac{w_0(x,\,\mu)}{2}\ . \tag{14}$$

Integrating by parts one readily finds that $\delta_n^{(\pi)}$ does not depend on μ and is given by

$$\delta_n^{(\pi)} = n(n+2)/6\ . \tag{15}$$

The expression for the coefficients $\delta_n^{(g)}$ can be found in a somewhat lengthy analytical form. The integrals in eq.(13) with the function $w_g(x,\,\mu)$ and $n \le 5$ are listed in the Appendix. Similar integrals for the perturbative coefficients $\delta_n^{(1)}$ can also, perhaps, be done analytically as a function of the mass ratio μ. However, judging by the expression[13, 14] for the function $w_1(x,\,\mu)$ and by the analytical expression for the dependence of the $O(\alpha_s)$ correction to the total rate[19], the resulting formulas should be prohibitively lengthy. For the practical purpose of analyzing experimental data it is sufficient however to have a table of these coefficients for values of μ around the approximate actual value $\mu \approx 0.3$. The numerical values of the coefficients $\delta_n^{(1)}$ and $\delta_n^{(g)}$ for $n \le 5$ are given in Tables 1 and 2. Since for each n these coefficients are slowly varying functions of μ, the tabular values can be used for an interpolation.

One can see from these numerical values and from eq.(15) that for $\alpha_s \approx 0.2$, $\mu_g^2/m_b^2 \approx 0.015$ and $\mu_\pi^2/m_b^2 \approx 0.015 - 0.025$ the perturbative correction to the ratia r_n is quite small as compared to the non-perturbative terms, and that each of the latter terms is within 10% - 15% range for $n = 5$, though the overall non-perturbative correction is significantly smaller due to a partial cancellation between the two terms.

3 Discussion

The estimates presented above illustrate that both the perturbative and the non-perturbative QCD corrections are sufficiently small and controllable in a number of ratia of moments of

the lepton spectrum in semileptonic B decays, which number is sufficient for a detailed experimental study of the kinematical parameters of these decays. A simple numerical inspection reveals that the ratia r_n are in fact sensitive to the quark mass difference $m_b - m_c$ rather than to the individual quark masses. Therefore it is quite likely, that the value of this mass difference can be determined with high precision from experimental data, while to separate each of the masses, one will have to rely on other types of analyses, e.g. on the existing determination of m_b from the Υ sum rules, or, possibly, on one from the inclusive spectrum of photons, generated by the process $b \to s\,\gamma$, which may become possible in a future development of the experiment [20].

One last remark is in order concerning a possible experimental measurement of the moments M_n in e^+e^- annihilation at the $\Upsilon(4S)$ resonance. Since the resonance is slightly above the $B\overline{B}$ threshold, the B mesons have momentum of about $0.3\,GeV$, and their measured lepton spectrum is slightly distorted by boost. However in order to account for this boost in the integral quantities like the moments M_n there is no need to transform the measured lepton energy distribution to the B rest frame. The reason is that a small boost the expressions for the moments in the laboratory frame remain valid after adding in quadrature the 'intrinsic' average momentum squared of the b quark in meson, μ_π^2 with that of the B meson in the laboratory frame $< \mathbf{p}^2 >$. This obviously amounts to replacing the μ_π^2 by the effective quantity

$$\overline{\mu}_\pi^2 = \mu_\pi^2 + < \mathbf{p}^2 > \quad . \tag{16}$$

One can notice that at the energy of the $\Upsilon(4S)$ resonance the effect of the boost: $< \mathbf{p}^2 > \approx 0.09 GeV^2$, is rather small in comparison with μ_π^2.

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A Appendix

The moments of the lowest-order energy distribution function, $I_n^{(0)} = \int_0^{1-\mu^2} w_0(x,\, \mu)\, x^n\, dx$, entering eq.(12), for $n \leq 5$ are given by the following expressions

$$I_0^{(0)} = 1 - 8\,\mu^2 + 8\,\mu^6 - \mu^8 - 12\,\mu^4\,\log(\mu^2) \tag{17}$$

$$I_1^{(0)} = \frac{7}{10} - \frac{15\,\mu^2}{2} - 12\,\mu^4 + 20\,\mu^6 - \frac{3\,\mu^8}{2} + \frac{3\,\mu^{10}}{10} - 6\,\mu^4\,\left(3 + \mu^2\right)\,\log(\mu^2) \tag{18}$$

$$I_2^{(0)} = \frac{8}{15} - \frac{36\,\mu^2}{5} - 26\,\mu^4 + 32\,\mu^6 + \frac{4\,\mu^{10}}{5} - \frac{2\,\mu^{12}}{15} - 8\,\mu^4\,\left(3 + 2\,\mu^2\right)\,\log(\mu^2) \tag{19}$$

$$I_3^{(0)} = \frac{3}{7} - 7\,\mu^2 - \frac{83\,\mu^4}{2} + \frac{85\,\mu^6}{2} + 5\,\mu^8 + \mu^{10} - \frac{\mu^{12}}{2} + \frac{\mu^{14}}{14} - 30\,\mu^4\,\left(1 + \mu^2\right)\,\log(\mu^2) \tag{20}$$

$$I_4^{(0)} = \frac{5}{14} - \frac{48\,\mu^2}{7} - \frac{291\,\mu^4}{5} + \frac{252\,\mu^6}{5} + 15\,\mu^8 - \mu^{12} + \frac{12\,\mu^{14}}{35} - \frac{3\,\mu^{16}}{70} - 12\,\mu^4\,\left(3 + 4\,\mu^2\right)\,\log(\mu^2) \tag{21}$$

$$I_5^{(0)} = \frac{11}{36} - \frac{27\,\mu^2}{4} - \frac{759\,\mu^4}{10} + \frac{329\,\mu^6}{6} + \frac{63\,\mu^8}{2} - \frac{7\,\mu^{10}}{2} - \frac{7\,\mu^{12}}{6} + \frac{9\,\mu^{14}}{10} - \frac{\mu^{16}}{4} + \frac{\mu^{18}}{36} - 14\,\mu^4\,\left(3 + 5\,\mu^2\right)\,\log(\mu^2)\;. \tag{22}$$

The first moments of the function w_g: $I_n^{(g)} = \int_0^{1-\mu^2} w_g(x,\, \mu)\, x^n\, dx$, necessary for calculation of the coefficients $\delta_n^{(g)}$, are given by the following expressions

$$I_0^{(g)} = -\frac{3}{2} + 4\,\mu^2 - 12\,\mu^4 + 12\,\mu^6 - \frac{5\,\mu^8}{2} - 6\,\mu^4\,\log(\mu^2) \tag{23}$$

$$I_1^{(g)} = -2 + \frac{5\,\mu^2}{3} - 4\,\mu^4 + 8\,\mu^6 - \frac{14\,\mu^8}{3} + \mu^{10} - 4\,\mu^2\,\log(\mu^2) \tag{24}$$

$$I_2^{(g)} = -\frac{104}{45} - \frac{8\,\mu^2}{3} + 25\,\mu^4 - \frac{40\,\mu^6}{3} - \frac{28\,\mu^8}{3} + \frac{16\,\mu^{10}}{5} - \frac{5\,\mu^{12}}{9} - 4\,\mu^2\,\left(2 - 3\,\mu^2 - \frac{10\,\mu^4}{3}\right)\,\log(\mu^2) \tag{25}$$

$$I_3^{(g)} = -\frac{53}{21} - \frac{42\,\mu^2}{5} + \frac{155\,\mu^4}{2} - \frac{95\,\mu^6}{2} - 25\,\mu^8 + 8\,\mu^{10} - \frac{73\,\mu^{12}}{30} + \frac{5\,\mu^{14}}{14} - 2\,\mu^2\,\left(6 - 15\,\mu^2 - 25\,\mu^4\right)\,\log(\mu^2) \tag{26}$$

$$I_4^{(g)} = -\frac{75}{28} - \frac{76\,\mu^2}{5} + \frac{1553\,\mu^4}{10} - 86\,\mu^6 - \frac{395\,\mu^8}{6} + 20\,\mu^{10} - \frac{73\,\mu^{12}}{10} + \frac{206\,\mu^{14}}{105} - \frac{\mu^{16}}{4} - 2\,\mu^2\,\left(8 - 27\,\mu^2 - 60\,\mu^4\right)\,\log(\mu^2) \tag{27}$$

$$I_5^{(g)} = -\frac{151}{54} - \frac{160\,\mu^2}{7} + \frac{1299\,\mu^4}{5} - \frac{1057\,\mu^6}{9} - \frac{455\,\mu^8}{3} + 49\,\mu^{10} - \frac{175\,\mu^{12}}{9} + \frac{103\,\mu^{14}}{15} - \frac{23\,\mu^{16}}{14} + \frac{5\,\mu^{18}}{27} - 4\,\mu^2\,\left(5 - 21\,\mu^2 - \frac{175\,\mu^6}{3}\right)\,\log(\mu^2)\;. \tag{28}$$

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m_c/m_b	$\delta_1^{(1)}$	$\delta_2^{(1)}$	$\delta_3^{(1)}$	$\delta_4^{(1)}$	$\delta_5^{(1)}$
0.25	0.0252	0.0669	0.1193	0.1787	0.2427
0.26	0.0226	0.0619	0.1120	0.1693	0.2312
0.27	0.0201	0.0571	0.1052	0.1603	0.2202
0.28	0.0178	0.0527	0.0987	0.1519	0.2098
0.29	0.0156	0.0485	0.0925	0.1438	0.1999
0.30	0.0135	0.0446	0.0867	0.1362	0.1904
0.31	0.0116	0.0408	0.0812	0.1289	0.1813
0.32	0.0099	0.0373	0.0760	0.1219	0.1726
0.33	0.0082	0.0340	0.0711	0.1153	0.1643
0.34	0.0066	0.0309	0.0663	0.1090	0.1563
0.35	0.0052	0.0280	0.0619	0.1029	0.1486

Table 1: Numerical values of the perturbative correction coefficients $\delta_n^{(1)}$ in eq.(11) for $n \leq 5$ and $\mu = m_c/m_b$ in the range from 0.25 to 0.35 .

m_c/m_b	$\delta_1^{(g)}$	$\delta_2^{(g)}$	$\delta_3^{(g)}$	$\delta_4^{(g)}$	$\delta_5^{(g)}$
0.25	-1.188	-2.419	-3.674	-4.94	-6.213
0.26	-1.187	-2.414	-3.664	-4.926	-6.195
0.27	-1.186	-2.411	-3.657	-4.915	-6.17
0.28	-1.185	-2.408	-3.651	-4.905	-6.165
0.29	-1.185	-2.406	-3.647	-4.898	-6.155
0.30	-1.185	-2.405	-3.644	-4.893	-6.147
0.31	-1.186	-2.406	-3.643	-4.89	-6.142
0.32	-1.187	-2.407	-3.644	-4.89	-6.14
0.33	-1.189	-2.409	-3.646	-4.891	-6.141
0.34	-1.192	-2.413	-3.65	-4.895	-6.144
0.35	-1.195	-2.417	-3.655	-4.901	-6.151

Table 2: Numerical values of the non-perturbative correction coefficients $\delta_n^{(g)}$ in eq.(11) for $n \leq 5$ and $\mu = m_c/m_b$ in the range from 0.25 to 0.35 .